Journal of Algebra **243,** 361–384 (2001)
doi:10.1006/jabr.2001.8854, available online at http://www.idealibrary.com on IDEAL®

The Classical Involution Theorem for Groups of Finite Morley Rank

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Communicated by Gernot Stroth

Received February 26, 1999

This paper gives a partial answer to the Cherlin–Zil'ber Conjecture, which states that every infinite simple group of finite Morley rank is isomorphic to an algebraic group over an algebraically closed field. The classification of the generic case of tame groups of odd type follows from the main result of this work, which is an analogue of Aschbacher's Classical Involution Theorem for finite simple groups.
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1. INTRODUCTION

This paper is a part of the ongoing project which aims to classify infinite simple groups of finite Morley rank, hence it deals with the following conjecture.

CHERLIN–ZIL'BER CONJECTURE. *An infinite simple group of finite Morley rank is isomorphic to an algebraic group over an algebraically closed field.*

Two obstacles have appeared so far in the attempt to prove the conjecture, namely bad groups and bad fields. A *bad group* is a connected non-solvable group of finite Morley rank all of whose proper definable connected subgroups are nilpotent. A *bad field* is a structure of finite Morley rank of the form $\langle F, +, \cdot, A\rangle$, where $\langle F, +, \cdot \rangle$ is an algebraically closed field and A is a proper infinite multiplicative subgroup of F^*. A group of finite Morley rank is *tame* if it does not interpret any bad groups or bad fields.

It is still not known if any bad groups or bad fields exist. However, a recent result by Wagner [29] seems to make the existence of bad fields of characteristic $p > 0$ rather unlikely.

[1] This work is a part of the author's Ph.D. thesis written at UMIST, England.

0021-8693/01 $35.00

A number of papers [1–4, 6, 11, 12, 23] have been written which apply the techniques of finite group theory to the model-theoretic context of groups of finite Morley rank and deal with the following tame version of the Cherlin–Zil'ber Conjecture.

TAME CONJECTURE. *A tame simple group of finite Morley rank is isomorphic to an algebraic group over an algebraically closed field.*

The attempt to prove the Tame Conjecture has proceeded by induction on Morley rank. Hence all the results of this project deal with a minimal counterexample to the Cherlin–Zil'ber Conjecture which is a K^*-group, that is, a group of finite Morley rank whose proper simple definable sections are simple algebraic groups.

The classification project has been divided into several smaller ones. The first division appeared in a work of Altınel et al. [2]. Later Jaligot proved a version of this result which does not use the tameness assumption.

FACT 1.1 [23]. *An infinite simple K^*-group G of finite Morley rank is of odd type or even type or degenerate type.*

Here, G is of *odd*, *even*, or *degenerate* type if its Sylow 2-subgroups are, respectively, abelian divisible-by-finite, definable of bounded exponent, or finite.

This paper is concerned with the following.

ODD TYPE CONJECTURE. *An infinite simple tame K^*-group of finite Morley rank and odd type is isomorphic to an algebraic group over an algebraically closed field of characteristic different from* 2.

Any classification theory of this nature requires an "identification theorem" at its final stage. In the case of "generic" groups of even type the result of [10] identifies them with simple algebraic groups over algebraically closed fields of characteristic 2. The purpose of this work is to prove Theorem 1.3, which is an "identification theorem" for "generic" groups of odd type.

According to the Trichotomy Theorem of Borovik in [11], which is another division in the project, the groups which satisfy the conditions of Theorem 1.3 constitute the class of "generic" groups of odd type. All technical terms are explained in the next section.

FACT 1.2 [11]. *If G is an infinite simple tame K^*-group of finite Morley rank and of odd type, then one of the following occurs*:

(i) *G has a classical involution and satisfies the B-conjecture.*

(ii) $n(G) \le 2$.

(iii) *G has a proper 2-generated core.*

In the present paper, "large" groups which fall into the first category of the Trichotomy Theorem are classified without the tameness assumption. This result is analogous to that of the famous paper by Aschbacher [8] on finite simple groups.

For any *classical involution* $z \in G$, by definition there exists a subgroup K (called the *classical subgroup*) which is isomorphic to $SL_2(\mathbb{F})$ for an algebraically closed field $\mathbb{F}$ and satisfies $z \in K \trianglelefteq C_G^o(z)$. Following the geometric ideas of Aschbacher [8], one can form a graph $\mathscr{D}$ with the vertex set $\{K^g : g \in G\}$ where two conjugates of K are joined by an edge if they commute elementwise.

THEOREM 1.3. *Let G be a simple K^*-group of finite Morley rank and odd type which satisfies the B-conjecture and contains a classical involution. Assume also that the graph $\mathscr{D}$ is connected. Then G is isomorphic to one of the following groups*: $PSL_n(\mathbb{F})$ *for* $n \geq 5$. $PSp_{2n}(\mathbb{F})$ *for* $n \geq 3$, $PSO_n(\mathbb{F})$ *for* $n \geq 9$, $E_6(\mathbb{F})$, $E_7(\mathbb{F})$, $E_8(\mathbb{F})$, *and* $F_4(\mathbb{F})$, *where* $\mathbb{F}$ *stands for an algebraically closed field of characteristic different from* 2.

The simple algebraic groups which are not covered by this theorem are as follows: $PSL_2(\mathbb{F})$, $PSL_3(\mathbb{F})$, $PSL_4(\mathbb{F}) \cong PSO_6(\mathbb{F})$, $PSO_5(\mathbb{F}) \cong PSp_4(\mathbb{F})$, $PSO_7(\mathbb{F})$, $PSO_8(\mathbb{F})$, and $G_2(\mathbb{F})$. The following is expected for the case when $\mathscr{D}$ is disconnected.

(1) If $n(G) \geq 3$, then $G \cong PSO_n(\mathbb{F})$ for $6 \leq n \leq 8$.

(2) If $n(G) \leq 2$, then $G \cong PSL_3(\mathbb{F})$ or $G_2(\mathbb{F})$ or $PSO_5(\mathbb{F}) \cong PSp_4(\mathbb{F})$.

Note that $PSL_2(\mathbb{F})$ is the only simple algebraic group of characteristic different from 2 which does not contain a classical involution. Algebraic groups over algebraically closed fields of characteristic 2 do not contain any classical involutions either, hence the study of such groups requires completely different techniques.

The method used in the proof of Theorem 1.3 is to identify the classical subgroups of G with the long root SL_2-subgroups of the target simple algebraic group. The final identification is done by a version of the Curtis–Phan theorem [18, 25] due to Lyons [20], which describes the generating relations between the fundamental root SL_2-subgroups of universal Chevalley groups.

2. DEFINITIONS

All definitions related to groups of finite Morley rank in general can be found in [13].

Assume that G is a group of finite Morley rank. The group G is called:

- *tame*, if G does not interpret any bad groups or bad fields;

- a *K-group*, if every infinite simple definable and connected section of the group is an algebraic group over an algebraically closed field;
- a *K^*-group*, if every proper definable subgroup of G is a K-group;
- of *odd type*, if its Sylow 2-subgroups are abelian divisible-by-finite;
- of *even type*, if its Sylow 2-subgroups are definable and of bounded exponent;
- of *degenerate type*, if its Sylow 2-subgroups are finite.

For a finite elementary abelian 2-group E, its *2-rank* $\mathrm{m}(E)$ is the minimal number of generators of E. If H is a (not necessarily definable) subgroup of a group of finite Morley rank and odd type, then set

$$\mathrm{m}(H) \coloneqq \max\{\mathrm{m}(E) : E \text{ is a finite elementary abelian 2-subgroup of } H\}.$$

If K is a definable subgroup of a group of finite Morley rank and odd type and if S is a Sylow 2-subgroup of K, then the *normal rank* $\mathrm{n}(K)$ is

$$\mathrm{n}(K) \coloneqq \max\{\mathrm{m}(E) : E \text{ is a normal finite elementary abelian subgroup of } S\}.$$

Since Sylow 2-subgroups are conjugate in a group of finite Morley rank (Fact 4.4), the normal rank is well-defined for definable subgroups of groups of finite Morley rank.

If G is of odd type and S is a Sylow 2-subgroup of G, then the definable closure of the group generated in G by all normalizers $N_G(U)$ of all subgroups $U \leq S$ with $\mathrm{m}(U) \geq 2$ is called the *2-generated core* of G.

A 2-subgroup S of a group of finite Morley rank and odd type contains, by definition, a subgroup S° of finite index which is a *Prüfer 2-group*, that is, a direct product of finitely many copies of the quasicyclic group $\mathbb{Z}_{2^\infty}$. The number of copies is called the *Prüfer 2-rank* of S and is denoted by $\mathrm{pr}(S)$. For a definable group H, $\mathrm{pr}(H)$ is the maximum of the Prüfer ranks of 2-subgroups in H. It is easy to see that $\mathrm{n}(H) \geq \mathrm{pr}(H)$.

The *Fitting subgroup* $F(G)$ is generated by the normal nilpotent subgroups of G. It is known that $F(G)$ is a definable nilpotent subgroup ([24] or Theorem 7.3 in [13]).

A group H is called *quasi-simple* if $H' = H$ and $H/Z(H)$ is simple and non-abelian. A quasi-simple subnormal subgroup of G is called a *component* of G. The product of all components of G is called the *layer* of G and is denoted by $L(G)$, and $E(G)$ stands for $L(G)^\circ$. It is known (see Lemmas 7.6 and 7.10 in [13]) that G has finitely many components and that they are definable and normal in $E(G)$.

The group G is said to satisfy the *B-conjecture* if, for any involution $z \in G$,

$$C_G^\circ(z) = F(C_G^\circ(z))E(C_G(z)).$$

For an arbitrary involution $z \in G$, a component A of $C_G(z)$ is called *intrinsic* if $z \in Z(A)$. An involution z is called *classical* if its centralizer $C_G(z)$ contains a *classical component*, that is, an intrinsic component isomorphic to $SL_2(\mathbb{F})$ for an algebraically closed field $\mathbb{F}$. A classical component of $C_G(z)$ is also referred to as a *classical subgroup* of G.

3. GENERALITIES ON ALGEBRAIC GROUPS

3.1. *Model Theory of Algebraic Groups*

When we speak about an algebraic group, we mean the group of points over an algebraically closed field K of a linear algebraic group defined over K. An algebraic group G over an algebraically closed field $\mathbb{F}$ is a group of finite Morley rank. If G is simple, then G contains a Borel subgroup B (that is, a maximal connected solvable subgroup) which is not nilpotent. Since the definable closure of a solvable group is solvable (Corollary 5.38 in [13] or [32]) and that of a connected group is connected (Lemma 5.40 in [13]), B is definable in the group structure of G. Hence the field $\mathbb{F}$ is interpretable in G as a pure group by Zil'ber's famous theorem (Corollary 9.10 in [13]), and all Zariski closed subgroups of G are definable in the pure group structure of G.

Note however that we consider algebraic groups with a possibly richer structure inherited from the definable universe $\mathscr{U}$, where these groups live. In particular, we have two different concepts of "closed" subgroups, Zariski closed and definable.

The term "definable" always refers to definability in some ranked universe $\mathscr{U}$ (see [13] for definitions and discussion). A definable (in our sense) subset of a group G is not necessarily definable in G as a pure group.

Therefore if G is a simple algebraic group, Zariski closed subgroups are definable and the Zariski closure $\overline{H}$ of a subgroup $H < G$ contains the definable closure $d(H)$.

The reader has to keep in mind that we also have two versions of the connected component of the identity, the definable connected component and Zariski closed component, the latter containing the former.

However, the use of these concepts in the present paper leaves no room for confusion.

3.2. *Relevant Structural Facts*

Most of the facts can be found easily in the standard references [15, 16, 21].

The symbol for the field $\mathbb{F}$ will be skipped in SL_2 will be used instead of $SL_2(\mathbb{F})$, etc.

First note that a connected algebraic group G is called *simple* if it has no proper normal connected and closed subgroups. Such a group turns out to have a finite center, the quotient group being simple as an abstract group.

Now fix a maximal torus T in G and denote the corresponding root system by Φ, and for each $\alpha \in \Phi$ denote the corresponding root subgroup by X_α. The subgroup $\langle X_\alpha, X_{-\alpha} \rangle$ is known to be isomorphic to SL_2 or PSL_2 and is called a *root SL_2-subgroup*.

If G is simple, the roots can have at most two different lengths, and the terms "short root SL_2-subgroup" and "long root SL_2-subgroup" have the obvious meanings. Long root SL_2-subgroups are always isomorphic to SL_2, while short root SL_2-subgroups in adjoint type B_n-groups are isomorphic to PSL_2.

An algebraic group is generated by its root SL_2-subgroups. In a simple algebraic group, all long root SL_2-subgroups are conjugate to each other, and similarly all short root SL_2-subgroups are conjugate to each other.

By the description of the centralizers of involutions in a simple algebraic group G of characteristic different from 2 (see [22] and compare with [8]) that is not PSL_2, it easily follows that the long root SL_2-subgroups are classical subgroups of G. Conversely, with the exception of G_2, a classical subgroup in G is conjugate to a long root subgroup. In particular, the classical subgroups form one conjugacy class in G, unless $G = G_2$.

In the group G_2, the centralizer of an arbitrary involution is isomorphic to $SL_2 * SL_2$, central product with amalgamated centers, where one factor corresponds to a long root SL_2-subgroup and the other to a short root SL_2-subgroup (so the factors are not conjugate). Now note that both factors are classical subgroups corresponding to the involutions in their centers. Hence G_2 has two conjugacy classes of classical subgroups.

The most important fact for our purposes is the following property of classical subgroups in simple algebraic groups; this property immediately follows from the characterization of these groups as long root SL_2-subgroups. Note that if G is a simple algebraic group, z is a classical involution in G, and K is a classical subgroup with $z \in Z(K)$, then z belongs to a maximal torus of K and hence z lies in a maximal torus, say T, of G. Then the connected group $T < C_G(z)^\circ$ normalizes a component K of $C_G(z)^\circ$, and hence TK is a closed reductive subgroup containing a maximal torus T. The structure of such subgroups is well-known [26], and, in particular, K is a long root SL_2-subgroup.

A slight modification of the same argument yields the following more general fact.

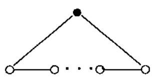
FACT 3.1. *Let G be a simple algebraic group over an algebraically closed field of characteristic different from 2, and let T be a maximal torus in G.*

Assume also that there are subgroups L and M which are isomorphic to $(P)SL_2$ and are normalized by T. Then either $[L, M] = 1$ or $\langle L, M\rangle$ is a simple algebraic group of one of the types A_2, B_2, or G_2, and in the latter case $G = \langle L, M\rangle$ is of type G_2. Moreover, in all cases, L and M are embedded in $\langle L, M\rangle$ as root $(P)SL_2$-subgroups.

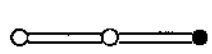
3.3. *Extended Dynkin Diagrams*

Our main computational tool is the following list of the extended Dynkin diagrams for simple algebraic groups. More information can be found in [16]. The black node corresponds to the negative of the highest root. Note that the highest root is a long root and the corresponding long root SL_2-subgroup is a classical subgroup.

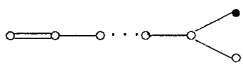
$\widetilde{A_n}$



$\widetilde{B_2} = \widetilde{C_2}$

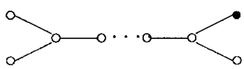


$\widetilde{B_n}$

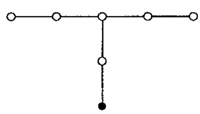


$\widetilde{C_n}$

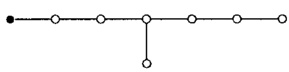
$\widetilde{D_n}$



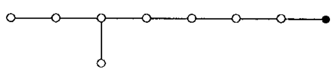
$\widetilde{E_6}$



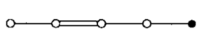
$\widetilde{E_7}$



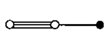
$\widetilde{E_8}$



$\widetilde{F_4}$



$\widetilde{G_2}$



4. USEFUL FACTS

In the model-theoretic literature, the following fact is often referred as Zil'ber's Indecomposability Theorem.

FACT 4.1 [31]. *A subgroup of a group of finite Morley rank which is generated by some definable connected subgroups is also definable and connected and is the setwise product of finitely many of them.*

Proof. See Corollary 5.28 in [13]. ∎

For a simple algebraic group G, $\mathrm{Inn}(G)$ stands for the group of inner automorphisms, and $\Gamma(G)$ stands for the group of graph automorphisms of G which preserve the lengths of the roots.

FACT 4.2 [13, Theorem 8.4]. *Let $\mathcal{G}$ be a group of finite Morley rank of the form $G \rtimes H$, where G and H are definable. Also assume that G is an infinite simple algebraic group over an algebraically closed field and $C_H(G) = 1$. Then one can view H as a subgroup of the group of automorphisms of G, and moreover $H \leq \mathrm{Inn}(G)\Gamma(G)$.*

FACT 4.3 [5]. *Suppose G is a group of finite Morley rank, $G = G'$, and $G/Z(G)$ is a simple algebraic group over an algebraically closed field and is of finite Morley rank; then $Z(G)$ is finite and G is also algebraic.*

FACT 4.4 [14]. *In a group of finite Morley rank, Sylow 2-subgroups are conjugate.*

Proof. See Theorem 10.11 in [13]. ∎

4.1. *On K-Groups and K^*-Groups*

In this section, G will stand for a simple K^*-group of odd type which satisfies the B-conjecture.

For any involution t in G, the K^*-assumption ensures that $C_G(t)$ is a K-group; moreover, $C_G^\circ(t) = F(C_G^\circ(t))E(C_G(t))$ by the B-conjecture. Actually, one can give a more precise description of $C_G^\circ(t)$. Being a K-group, $C_G(t)$ has components which are centrally isomorphic to algebraic groups over algebraically closed fields. Now Fact 4.3 implies that each component is algebraic. These arguments can be summarized as follows.

LEMMA 4.5. *For any involution $t \in G$,*

$$C_G^\circ(t) = D \cdot L_1 \cdots L_r,$$

where $D = F(C_G^\circ(t))$ and each L_i is a simple algebraic group over an algebraically closed field.

LEMMA 4.6. *Assume H is a K-group of finite Morley rank satisfying*

$$H^\circ = F(H^\circ)E(H),$$

t is an involution in H, and L is a classical component in $E(C_H(t))$. Then there exists a component $M \trianglelefteq E(H)$ such that $L \leq M$.

Proof. This is a version of Lemma 6.16 in [11] and Lemma 2.7(2) in [7]. ∎

A 4-*group* is an elementary abelian group of order 4.

FACT 4.7 [11, Lemma 4.5]. *If* $\mathrm{n}(G) \geq 3$, *then a Sylow* 2-*subgroup S of G is* 2-*connected*; *that is*, *for any two* 4-*subgroups* V, $W \leq S$, *there exists a sequence of* 4-*subgroups in* S

$$V = V_1, V_2, \ldots, V_n = W$$

such that $[V_i, V_{i+1}] = 1$ *for all* $i = 1, \ldots, n-1$.

For any 4-subgroup $V \leq G$, set $\Gamma_V(G) := \langle C_G^\circ(v) \mid v \in V \setminus \{1\} \rangle$. By Fact 4.1, $\Gamma_V(\Gamma)$ is a definable connected subgroup of G.

FACT 4.8 [11, Theorem 5.14]. *Let H be a K-group of finite Morley rank of odd type and $V \leq H$ a* 4-*subgroup of H*; *then* $\Gamma_V(H) = H^\circ$.

COROLLARY 4.9. *If V, W are* 4-*subgroups of G and* $[V, W] = 1$, *then* $\Gamma_V(G) = \Gamma_W(G)$.

Proof. Note that by Fact 4.8, for any $w \in W \setminus \{1\}$,

$$C_G^\circ(w) = \Gamma_V(C_G^\circ(w)) \leq \Gamma_V(G).$$

Hence $\Gamma_W(G) \leq \Gamma_V(G)$, and thus $\Gamma_W(G) = \Gamma_V(G)$ by symmetry. ∎

5. LYONS THEOREM

The following theorem, due to Lyons, is a generalization of the Curtis–Phan Theorem [18, 25]. In [20] it is formulated in the version for finite Chevalley groups, but its proof is valid, without any changes, for Chevalley groups over arbitrary fields.

The definition of a defining amalgam can be found in [19, Section 29].

FACT 5.1 [20, Theorem 1.24.2]. *Let G be the universal version of a Chevalley group over an algebraically closed field with root system Φ, system of simple roots Δ, and root groups X_α, $\alpha \in \Phi$. Suppose that Φ has rank at least 3. For each $J \subseteq \Delta$ let G_J be the subgroup of G generated by all root* *subgroups X_α, $\pm\alpha \in J$. Let D be the set of all subsets of Δ with at most two elements, and partially order D by inclusion. Then $\{G_J \mid J \in D\}$ is a defining G-amalgam.*

According to [19, Proposition 29.1], this result translates into the following:

FACT 5.2 [19]. *Let G be a group, $\mathbb{F}$ an algebraically closed field, and I one of the connected Dynkin diagrams of the simple algebraic groups of Tits rank at least 3. Let $\{K_i : i \in I\}$ be a collection of subgroups of G centrally isomorphic to $PSL_2(\mathbb{F})$ indexed by I, and let $T_i < K_i$ be a maximal torus in K_i for $i \in I$. Also assume that the following statements hold.*

(1) $G = \langle K_i : i \in I \rangle$.

(2) $[T_i, T_j] = 1$ *for all* $i, j \in I$.

(3) $[K_i, K_j] = 1$, *if* $i \neq j$ *and* (i, j) *is not an edge in* I.

(4) $G_{ij} = \langle K_i, K_j \rangle$ *is centrally isomorphic to* $PSL_3(\mathbb{F})$ *if* (i, j) *is a single edge in* I.

(5) $G_{ij} = \langle K_i, K_j \rangle$ *is centrally isomorphic to* $PSp_4(\mathbb{F}) \cong PSO_5(\mathbb{F})$ *if* (i, j) *is a double edge in* I. *Moreover, in that case, if* r_i *and* r_j *are involutions in* $N_{K_i}(T_iT_j)$ *and* $N_{K_j}(T_iT_j)$, *then the order of the element* r_ir_j *in* $N_{G_{ij}}(T_iT_j)/T_iT_j$ *is* 4.

(6) K_i, K_j *are root* SL_2*-subgroups of* G_{ij} *corresponding to the maximal torus* T_iT_j *of* G_{ij}.

Then there is a homomorphism from the corresponding simply connected simple algebraic group $\tilde{G}$ of type I over $\mathbb{F}$ onto G, under which the root SL_2-subgroups $\tilde{K}_i$ for some simple root system of $\tilde{G}$ correspond to the subgroups K_i.

Proof. First note that $\tilde{G}$, root SL_2-subgroups $\tilde{K}_i$ and the subgroups $\tilde{K}_{ij} = \langle \tilde{K}_i, \tilde{K}_j \rangle$, being the Levi factors of the corresponding parabolic subgroups, are all simply connected (see, for example [28, p. 75]). Condition (5) ensures that when $G_{ij}/Z(G_{ij}) \cong PSp_4(\mathbb{F}) \cong PSO_5(\mathbb{F})$, the groups K_i and K_j are embedded in the group G_{ij} as root SL_2-subgroups corresponding to roots of different lengths. Hence for any system of $(P)SL_2$-subgroups K_i satisfying the conditions (1)–(6), there is a morphism of amalgams

$$\left\{\tilde{K}_i, \tilde{K}_j, \tilde{K}_{ij}\right\} \to \{K_i, K_j, K_{ij}\}.$$

This property and Fact 5.1 together satisfy the requirements of Proposition 29.1 in [20], and hence there is a surjective homomorphism $\tilde{G} \to G$.

The final remark is about distinguishing groups of type B_n and C_n. Let us label the Dynkin diagram so that the nodes 1 and 2 are connected by a double edge, while 1 is connected only to 2 and 2 is connected to 1 and 3. In the case of B_n, the group K_1 is a short root SL_2-subgroup and its factor in $G_{12}/Z(G_{12})$ is isomorphic to $PSL_2(\mathbb{F})$, whereas in the case of C_n, the group K_1 is isomorphic to $SL_2(\mathbb{F})$. ∎

6. REFLECTION GROUPS

A *reflection* is a linear semisimple transformation of finite order with exactly one eigenvalue which is not 1.

THEOREM 6.1. *Let W be a finite group which has a faithful irreducible representation of dimension $n \geq 3$ over the complex numbers $\mathbb{C}$ and faithful irreducible representations* (*possibly of different dimensions*) *over prime fields $\mathbb{F}_p$ for almost all prime numbers p, such that, for each of these representations, W is generated by reflections of order 2. Then W is a Weyl group* (*a finite crystallographic reflection group*) *and is one of the groups A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4.*

Proof. Only a general description of a rather routine case-by-case proof will be given.

All irreducible complex reflection groups are classified in [27]; another, more convenient and modern proof is given in [17].

Note that one can assume, without loss of generality, that $p > |W|$. As it follows from the classification (see the character tables and other results on representation theory in [9]), lifts of reflection representations from $\mathbb{F}_p$ to $\mathbb{C}$ are equivalent to a unique complex reflection representation of W. Hence all representations have the same dimension $n \geq 3$. After this observation, all types of reflection groups other than Weyl groups can be excluded for one of the following reasons.

- If W contains a reflection of order $m > 2$, then this element is a reflection over $\mathbb{F}_p$ for almost all p, which is an obvious contradiction, since one can find, using Dirichlet's theorem about primes in arithmetic progressions, infinitely many prime numbers p such that m does not divide $p - 1$.

- If $|Z(W)| > 2$, then since $Z(W)$ consists of scalar matrices, $|Z(W)|$ divides $p - 1$ for almost all primes p; this is another contradiction to Dirichlet's Theorem.

- If $|W|$ is divisible by a power of a prime which does not divide $|GL_n(\mathbb{F}_p)|$ for some (and, therefore, for infinitely many) p, then again this is an easy application of the same arguments with primes in arithmetic progressions. ∎

7. ASCHBACHER'S GEOMETRIC ANALYSIS

The lemmas of this section were adapted, with appropriate changes, from Aschbacher [8]. From now on, G will be a group which satisfies the hypothesis of Theorem 1.3 (that is, a simple K^*-group of finite Morley rank and odd type, satisfying the B-conjecture and containing a classical involution, and the graph $\mathscr{D}$ of G is connected), z a classical involution of G, $\mathbb{F}$ an algebraically closed field of characteristic different from 2, and $K \cong SL_2(\mathbb{F})$ a classical subgroup in $C_G(z)$. In fact the connectedness of $\mathscr{D}$ is not necessary for the following two lemmas.

LEMMA 7.1. *Set $J := K^g$ for $g \in G$ such that $K \neq K^g$.*

(1) *The subgroup K has non-abelian Sylow 2-subgroups, z is the unique involution in K and K is subnormal in $C_G(z)$.*

(2) *If $J^z = J$, then there exists a maximal torus in J which normalizes K.*

(3) *If $K \cap J = \langle z \rangle$, then $[K, J] = 1$.*

(4) *If a Sylow 2-subgroup of K normalizes J, then $[K, J] = 1$.*

(5) *Any 2-element of $J \setminus \langle z^g \rangle$ which centralizes z normalizes K.*

Proof. (1) This is well known.

(2) Note that $C_J(z)$ is either J or a maximal torus of J; hence $C_J(z)$ is connected in either case and acts on $C_G(z)$. Since an action of a connected group on a finite set is trivial, $C_J(z)$ normalizes all components of $C_G(z)$, and in particular $C_J(z)$ normalizes K.

(3) If $K \cap J = \langle z \rangle$, then K and J are distinct intrinsic components in $E(C_G(z))$, hence they commute.

(4) Let S be the Sylow 2-subgroup of K which normalizes J, note that $z \in S$, hence $J^z = J$, so by Part (2), $C_J(z)$ normalizes K, in particular $K^{z^g} = K$, since $S \le C_K(z^g)$, $C_K(z^g) = K$, and therefore $K \le E(C_G(z^g))$. Since J is a classical subgroup of $C_G(z^g)$, $[K, J] = 1$.

(5) Let a be a 2-element in $J \setminus \langle z^g \rangle$ such that $[a, z] = 1$; then $[z^g, z] = 1$ and hence $[J^z, z^g] = 1$, which proves that J^z is intrinsic in $E(C_G(z^g))$. Hence either $J = J^z$ or $[J, J^z] = 1$. The latter yields a contradiction since $a \notin Z(J)$. Hence $J = J^z$, and Part (2) proves that $a \in C_J(z)$ normalizes K. ∎

Let Ω denote the collection of all conjugates of K in G. Construct a graph $\mathscr{D}$ with the vertex set Ω such that $J \in \Omega$ is joined to $L \in \Omega$ if and only if $[J, L] = 1$. For any $\alpha \subseteq \Omega$, set

$$\Omega(\alpha) := \{L \in \Omega \mid [J, L] = 1 \text{ for all } J \in \alpha\}.$$

To simplify notation, $\Omega(K)$ will be used to denote $\Omega(\{K\})$.

Let $\mathscr{D}^c$ stand for the complementary graph of $\mathscr{D}$ and note that G is transitive on Ω. Note also that $G = \langle \Omega \rangle$, since G is simple.

LEMMA 7.2. *The complementary graph $\mathscr{D}^c$ is connected.*

Proof. Assume the converse and let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two distinct connected components in $\mathscr{D}^c$. Suppose that $K_i \in \Omega$, K_i lies in $\mathscr{C}_i$, and let H_i be the group generated by the subgroups lying in $\mathscr{C}_i$ for $i = 1, 2$. By the construction of $\mathscr{D}^c$ and the fact that $G = \langle \Omega \rangle$, $H_i \trianglelefteq G$. However, G is transitive on Ω and thus there exists $g \in G$ such that $K_1^g = K_2 \leq H_2$, which is a contradiction. Hence $\mathscr{D}^c$ is connected. ∎

Let $\mathscr{O}(K)$ be the set of all connected components of $\Omega(K)^c$. The connectedness of $\mathscr{D}$ guarantees that $\Omega(K) \neq \phi$. Note that the subgroups which belong to different connected components of $\mathscr{O}(K)$ commute with each other.

Moreover, the subgroup which is generated by an element of $\mathscr{O}(K)$ in G is in fact generated by some conjugates of K, thus it is definable and connected by Fact 4.1. Hence the Morley rank of such a subgroup is well defined. Let Π be an element of $\mathscr{O}(K)$ such that

$$\mathrm{RM}(\langle \Pi \rangle) = \max\{\mathrm{RM}(\langle \Sigma \rangle) \mid \Sigma \in \mathscr{O}(K)\},$$

where RM stands for the Morley rank. In particular, $\langle \Pi \rangle$ is definable, connected, and generated by finitely many subgroups from Ω by Fact 4.1.

Note that the groups generated by the components of $\Omega(K)^c$ are subgroups of $E(C_G(z))$.

A proper subset α of Ω is called a *clump* of $\mathscr{D}$ if $\Omega(A) \subseteq \alpha \cup \Omega(\alpha)$ for any $A \in \alpha$.

LEMMA 7.3. *If $\beta := \Omega(\Pi)$, then β is a clump of $\mathscr{D}$ and $\Omega(\beta) = \Pi$.*

Proof. Let $P \in \Omega(\Pi)$ and $B \in \Omega(P) \setminus \Pi$. Note that B is adjacent in $\mathscr{D}$ to all subgroups of Π because otherwise B would be adjacent in $\mathscr{D}^c$ to at least one subgroup of Π and hence $\Omega(P)^c$ would have a connected component containing Π and B. Note that B and K are conjugate and hence this contradicts the maximality of the rank of $\langle \Pi \rangle$. Thus $B \in \Omega(\Pi)$ and hence

$$\Omega(P) \subseteq \Pi \cup \Omega(\Pi) \subseteq \Omega(\Omega(\Pi)) \cup \Omega(\Pi)$$

proves that $\Omega(\Pi)$ is a clump of $\mathscr{D}$.

To prove that $\Omega(\beta) = \Pi$, take an arbitrary $B \in \Omega(\beta) \setminus \Pi$. In particular, $B \in \Omega(P) \subseteq \Pi \cup \Omega(\Pi)$ for some $P \in \Omega(\Pi)$, thus $B \in \Omega(\Pi) \cup \Omega(\Omega(\Pi))$, which shows that B is adjacent in $\mathscr{D}$ to itself; that is, $[B, B] = 1$, which is a contradiction. Thus $\Omega(\beta) \setminus \Pi = \phi$. Since $\Pi \subseteq \Omega(\beta)$, the result follows. ∎

LEMMA 7.4. *With the above notation and assumptions, β is a set of imprimitivity for the action of G on Ω; that is, for any $g \in G$, either $\beta^g = \beta$ or $\beta^g \cap \beta = \phi$.*

Proof. It is enough to prove the following claim.

CLAIM. *If $\beta \cap \beta^g \neq \phi$ for some $g \in G$, then $\beta^g \subseteq \beta$.*

To prove the claim, let $A \in \beta \cap \beta^g$. Note that $\Pi \subseteq \Omega(A) \subseteq \beta^g \cup \Omega(\beta^g)$, since $A \in \beta = \Omega(\Pi)$ and β^g is a clump. If $\Pi \cap \Omega(\beta^g) \neq \phi$, then $\Pi \cap \beta^g = \phi$. Therefore $\Pi \cap \Omega(\beta^g) \neq \phi$ implies that $\Pi \subseteq \Omega(\beta^g)$ and hence $\beta^g \subseteq \Omega(\Pi) = \beta$.

Now assume that $\Pi \cap \Omega(\beta^g) = \phi$; that is, $\Pi \subseteq \beta^g$. Since $\beta^g \neq \Omega$ and $\mathscr{D}$ is connected, $\Omega(\beta^g)$ is not empty, so one can take an element $B \in \Omega(\beta^g) \subseteq \beta$ to obtain

$$\beta^g \subseteq \Omega(B) \subseteq \beta \cup \Omega(\beta) = \beta \cup \Pi. \tag{1}$$

Now consider an element $C \in \Pi \subseteq \beta^g$; then

$$\Omega(C) \subseteq \beta^g \cup \Omega(\beta^g) \subseteq \Pi \cup \beta$$

by Eq. (1). Next consider an element $C \in \beta$ to get

$$\Omega(C) \subseteq \Omega(\Pi) \cup \Omega(\Omega(\Pi)) = \beta \cup \Pi.$$

The last two equations prove that the elements of $\Pi \cup \beta$ are connected only to the elements of $\Pi \cup \beta$ in $\mathscr{D}$; i.e., $\Pi \cup \beta$ is a connected component of $\mathscr{D}$. Then $\Omega = \Pi \cup \beta$ since $\mathscr{D}$ is connected. But then $\mathscr{D}^c$ must be disconnected, which contradicts Lemma 7.2 ∎

8. COMPONENT ANALYSIS

The previous section consisted almost entirely of arguments adapted from Aschbacher [8]. In the first four lemmas of this section, we start to deviate from the path forged by Aschbacher and exploit the specific properties of groups of finite Morley rank.

The assumptions and the notation of the previous section are valid in this section as well; that is, G is a group which satisfies the hypothesis of Theorem 1.3, z a classical involution of G, $\mathbb{F}$ an algebraically closed field of characteristic different from 2, and $K \cong SL_2(\mathbb{F})$ a classical subgroup in $C_G(z)$.

LEMMA 8.1. *If $\mathscr{D}$ is connected, then* $\mathrm{pr}(G) \geq 3$. *In particular,* $\mathrm{n}(G) \geq 3$.

Proof. First note that there exist $J_1, J_2 \in \Omega$ such that $[J_1, J_2] = 1$, since otherwise $\mathscr{D}$ would have no edges. Also, one can find a subgroup

$J_3 \in \Omega \setminus \{J_1, J_2\}$ such that J_3 is connected to either J_1 or J_2 by an edge. (Otherwise, $\{J_1, J_2\}$ would be a connected component in $\mathscr{D}$ and hence $\Omega = \{J_1, J_2\}$, that is, $\mathscr{D}^c$ would be disconnected.) Say J_3 is connected to J_2, that is, $[J_2, J_3] = 1$.

Set $J := \langle J_1, J_2, J_3 \rangle$. If $[J_1, J_3] = 1$, then obviously $\mathrm{pr}(J) = 3$. So assume that $[J_1, J_3] \neq 1$. But then $J_1, J_3 \in C_G^\circ(z(J_2))$ and hence they lie in the same component M of $C_G^\circ(z(J_2))$. The group M is a simple algebraic group by Lemma 4.5 and $[M, J_2] = 1$. Hence $\mathrm{pr}(\langle J_1, J_3 \rangle) = 2$ since M is an algebraic group and $J = \langle J_1, J_3 \rangle \times J_2$, thus $\mathrm{pr}(J) = 3$. Clearly $\mathrm{pr}(G) \geq \mathrm{pr}(J) = 3$; hence $\mathrm{n}(G) \geq 3$, and this proves the lemma. ∎

For a definable subgroup $H \leq G$, set $\Omega \cap H := \{J \in \Omega \mid J \leq H\}$. Then $\langle \Omega \cap H \rangle$ is definable and connected by Fact 4.1.

Note that, since the graphs $\mathscr{D}$ and $\mathscr{D}^c$ are connected, the set $\Omega \cap C_G(K)$ contains at least two different subgroups from Ω; therefore the group $\langle \Omega \cap C_G(K) \rangle$ contains an involution $z' \neq z$.

Recall that $\Gamma_V(G) = \langle C_G^\circ(v) \mid v \in V \setminus \{1\} \rangle$ for a 4-subgroup $V \leq G$.

LEMMA 8.2. *For an involution $z' \neq z$ and $z' \in C_G(K)$, set $V := \langle z, z' \rangle$. Then $\Gamma_V(G) = G$.*

Proof. First note that since $\mathscr{D}$ is connected, there exists a subgroup $J \in \Omega$ such that $[K, J] = 1$.

Set $W := \langle z, z(J) \rangle$, then one can choose a Sylow 2-subgroup of G containing V and W. Note that, by Lemma 8.1, Fact 4.7 can be applied to G to get that S is 2-connected. Now, by Corollary 4.9, $\Gamma_W(G) = \Gamma_V(G)$, and hence

$$C_G^\circ(J) = \Gamma_W(C_G^\circ(J)) \leq \Gamma_W(G) = \Gamma_V(G).$$

Since $\mathscr{D}$ is connected, $\Omega \subseteq \Gamma_V(G)$ and hence $G = \langle \Omega \rangle \leq \Gamma_V(G)$, which proves the lemma. ∎

LEMMA 8.3. *The group $\langle \Omega \cap C_G(z) \rangle^\circ$ has no components isomorphic to $G_2(F)$.*

Proof. Let N be a component of $\langle \Omega \cap C_G(z) \rangle^\circ$ isomorphic to $G_2(F)$.

First assume that there exists an involution $z' \in C_G(KN) \setminus \{z\}$; then, by Lemma 8.2, $\Gamma_V(G) = G$ where $V = \langle z, z' \rangle$. Also note that

$$N \trianglelefteq C^\circ_{C_G^\circ(z')}(z) \leq C_G^\circ(z).$$

Now Lemma 4.6 implies that there exists a component $M \trianglelefteq C_G^\circ(z')$ such that $N \leq M$. Note that z permutes the components of $C_G^\circ(z')$ and normalizes M, hence N is a component of $C_M(z)$. By Fact 4.2, the automorphism induced by z on M lies in $\mathrm{Inn}(M)\Gamma_M$. The involutory automorphisms of algebraic groups are well known and this is possible only if $M = N$; that is, N is a component of $C_G^\circ(z')$. Thus $\Gamma_V(G) \leq N_G(N) \neq G$ where $V := \langle z, z' \rangle$, contradicting Lemma 8.2.

This contradiction shows that $C_G(KN)$ has exactly one involution z, that is; $\langle \Omega \cap C_G(z) \rangle^\circ$ is either $K \times K^* \times N$ or $K \times N$, where K^* is a classical subgroup of G and $z = z(K^*)$. (Note that if there are three different classical subgroups, say K, K^*, K^{**}, as components of $E(C_G(z))$ with a common involution, then K^*K^{**} contains an involution different from z which lies in $C_G(KN)$.)

Let T be a maximal torus of $\langle \Omega \cap C_G(z) \rangle^\circ$; then K (and also K^*) is conjugate to some $J \in \Omega \cap N$ in $N_G(T)$. Let $W := N(T)/T$ and $W_0 \leq W$ be the subgroup of W generated by reflections. Thus

$$\mathbb{Z}_2 \times W(G_2(F)) = W(K) \times W(G_2(F)) < W_0 \trianglelefteq W.$$

Being the maximal torus of $\langle \Omega \cap C_G(z) \rangle^\circ$, T is of dimension 3 or 4 (depending on whether K^* exists or not). The group W acts on T, and the group W_0 acts on the elementary abelian p-subgroups of T for all $p \neq \mathrm{char}(\mathbb{F})$. In [30], the list of all finite linear groups generated by reflections of order 2 is given, and for $p \neq 3$, $\mathbb{Z}_2 \times W(G_2(F))$ cannot be a proper subgroup of such a group. Hence the result follows from this final contradiction. ∎

LEMMA 8.4. *Let Σ stand for a connected component of $\Omega(K)^c$. Then $\langle \Sigma \rangle$ is transitive on Σ.*

Proof. First note that $\langle \Sigma \rangle$ is a component in $C_G^\circ(z)$, hence it is a simple algebraic group by Lemma 4.5. Moreover, $\langle \Sigma \rangle$ is a simple algebraic group which is not $G_2(F)$ by Lemma 8.3. In such groups, the classical subgroups are conjugate to each other, as was remarked in Section 3.2; hence the result follows. ∎

The rest of this section is an adaptation of Aschbacher [8].

Recall that Π was defined to be an element of $\mathscr{O}(K)$ such that

$$\mathrm{RM}(\langle \Pi \rangle) = \max\{\mathrm{RM}(\langle \Sigma \rangle) \mid \Sigma \in \mathscr{O}(K)\}.$$

LEMMA 8.5. *Let $K^g \in \Pi$. Then $G = \langle \Pi, \Pi^g \rangle$.*

Proof. Recall that β stands for $\Omega(\Pi)$ and note that β^g is a clump, hence $\beta^g \subseteq \Omega(K) \subseteq \Pi \cup \beta$. However by Lemma 7.4, $\beta \cap \beta^g = \varnothing$ and thus $\beta^g \subseteq \Pi$. Now set

$$\Gamma = \{J \in \Omega \mid J \text{ is conjugate to } K \text{ or } K^g \text{ under } \langle \Pi, \Pi^g \rangle\}$$

to get $\Omega(K^g) \subseteq \beta^g \cup \Pi^g \subseteq \Pi \cup \Pi^g \subseteq \Gamma$ by the above argument and Lemma 8.4. Similarly, $\Omega(K) \subseteq \Gamma$. Since $[K, K^g] = 1$, $\Omega(K) \cup \Omega(K^g)$ is connected in $\mathscr{D}$ and it is easy to see that $\Omega(K) \cup \Omega(K^g)$ is a connected component of $\mathscr{D}$. Now $\mathscr{D}$ is connected and hence $\Gamma = \Omega$ and $G = \langle \Omega \rangle = \langle \Pi, \Pi^g \rangle$. ∎

Let X stand for $\langle \Pi \rangle$ and set $V(J) = \{J \in \Omega \mid z(J) = z(L)\}$ for any $J \in \Omega$, where $z(J)$ is the unique involution of J. Note that $K \in V(K)$ and $|V(K)|$ is independent of the choice of $K \in \Omega$.

LEMMA 8.6. (1) $C_G(KX)$ contains exactly one involution $z = z(K)$.
(2) $V(K) \setminus \{K\} = \Omega(K) \cap \beta$ and $|V(K)| \leq 2$.

Proof. (1) Assume $u \in C_G(KX)$, $u^2 = 1$, and $u \notin \langle z \rangle$, and let $K^g \in \Pi$. Set $\Gamma := \Omega \cap C(u)$, $A := \langle \Gamma \rangle$, and $\Sigma := (K^g)^A$; then $\Pi \subseteq \Sigma$. Assume that $K \in \Sigma$ to get a contradiction; then $K^a = K^g$ for some $a \in A$. But then $\Pi^g = \Omega(K^g) = \Pi^a \subseteq A$. Thus $G = \langle \Pi, \Pi^g \rangle = A$ and $u \in Z(G)$, which is a contradiction. Therefore $K \in \Gamma \setminus \Sigma$ and hence $\Pi = \Sigma$. Thus X is a component of $C_G^\circ(u) = A$ and commutes with other components of $C_G^\circ(u)$. In short, $C_G^\circ(u) \leq N_G(X)$. Now set $V := \langle u, z \rangle$; note that V is a 4-group and

$$\Gamma_V(G) := \left\langle C_G^\circ(v) \mid v \in V \setminus \{1\} \right\rangle \leq N_G(X) \neq G.$$

But this contradicts Lemma 8.2.

(2) It is clear that $V(K) \setminus \{K\} \subseteq \Omega(K) \cap \beta$. To see the converse, let $K^* \in \Omega(\Pi)$ and $[K, K^*] = 1$. Since both $z(K)$ and $z(K^*)$ centralize KX, Part (1) implies $z(K) = z(K^*)$.

To prove that $|V(K)| \leq 2$, assume the converse and take K, K_1, $K_2 \in V(K)$. Then $K_1 K_2 \cap C_G(KX)$ contains an involution which is not z; therefore $|V(K)| = 2$. ∎

COROLLARY 8.7. *The group $C_G^\circ(z)$ has either two components, K and X, or three components, K, K^*, and X, where $K^* \in \Omega$ and $z(K^*) = z(K)$. Moreover, in any case, X is a simple algebraic group which is not isomorphic to $G_2(\mathbb{F})$ and the Lie rank of X is at least* 2.

Proof. The first sentence is a direct consequence of Lemma 8.6 and X is not isomorphic to $G_2(\mathbb{F})$ by Lemma 8.3. To see that the Lie rank of X is at least 2, assume that it is 1, hence $X = K^g$ for some $g \in G$. Then $\Pi = \{K^g\}$, $\Pi^g = \{K\}$. By Lemma 8.5, $G = \langle \Pi, \Pi^g \rangle = K \times K^g$ and hence G is not simple. ∎

9. PROOF OF THE THEOREM

The idea of the proof is to construct, from the information about $C_G(z)$ which was obtained in the previous sections, the Weyl group and the root system of G and then to apply the Lyons Theorem.

9.1. *Natural Tori*

A *natural one-dimensional torus* of G is a maximal torus, in the sense of the theory of algebraic groups in one of the subgroups $SL_2 \cong L \in \Omega$. A *natural torus* is a product of pairwise-commuting natural one-dimensional tori.

LEMMA 9.1. *Let S be a Sylow 2-subgroup of G; $z_i = z(J_i) \in S^\circ$ a classical involution for some $J_i \in \Omega$; and T_i a maximal torus in J_i such that $T_i \cap S^\circ$ is a Sylow 2-subgroup of T_i for $i = 1, 2$. Then $[T_1, T_2] = 1$.*

Proof. Throughout this proof, let $i \in \{1, 2\}$. First note that S° is abelian, hence $S^\circ \le C_G(z_i)^\circ$. The classical subgroup J_i, being a component of $C_G(z_i)^\circ$, is normalized by S°. However, J_i does not have any rational outer automorphisms, since $J_i \cong SL_2$; thus $S^\circ = C_{S^\circ}(J_i) * (S^\circ \cap J_i)$ and hence $[S^\circ, T_i] = 1$. Note that $S^\circ \cap T_i = S^\circ \cap J_i$, therefore $[T_1, S^\circ \cap J_2] = 1$ and, in particular, $[T_1, z_2] = 1$; and hence T_1 normalizes J_2. By an argument like the one above, $T_1 = C_{T_1}(J_2) * (T_1 \cap J_2)$ and thus $[T_1, T_2] = 1$. ∎

From this lemma it is obvious that there is a correspondence between the maximal, with respect to inclusion, natural tori of G, and the Sylow 2-subgroups in G. More precisely, if S is a Sylow 2-subgroup of G, then the *maximal natural torus associated with* S is defined to be the group generated by all one-dimensional tori D such that $D \cap S^\circ$ is a Sylow 2-subgroup in D; conversely, any maximal natural torus of G is associated with some Sylow 2-subgroup of G.

LEMMA 9.2. (1) *All maximal natural tori in G are conjugate.*

(2) *If T is a maximal natural torus, then $N_G(T)$ controls 2-fusion in T; that is, if X and Y are two subsets of T consisting of 2-elements and $X^g = Y$ for some $g \in G$, then $X^h = Y$ for some $h \in N_G(T)$.*

Proof. (1) The conjugacy of maximal natural tori follows from the previous remark and the 2-Sylow Theorem for groups of finite Morley rank (Fact 4.4).

(2) Now let T be a maximal natural torus of G, and let $X, Y \subseteq T$ consist of 2-elements such that, for some $g \in G$, $X^g = Y$. Let S be the Sylow 2-subgroup corresponding to T; then $S^\circ \cap T$ is a Sylow 2-subgroup of T. Now note that $X, Y \subseteq S^\circ$, also $Y \subseteq S^{\circ g}$ and hence S°, $S^{\circ g} \le C_G(Y)$.

By Fact 4.4, there exists $c \in C_G(Y)$ such that $S^\circ = S^{\circ\, gc}$. Hence $gc \in N_G(S^\circ) = N_G(T)$; also note that $X^{gc} = Y$. ∎

9.2. *Root Subgroups*

From now on, T will be a maximal natural torus which contains the classical involution z. Note that the proof of Lemma 8.1 can be repeated to show that $\mathrm{pr}(T) \geq 3$.

Notation. If Y is a definable connected subgroup such that $T \cap Y$ is a maximal natural torus in Y, then set $T_Y := T \cap Y$. If R is a one-dimensional torus, then $z(R)$ is the unique involution of R.

Now consider the collection $\Gamma' := \{L \in \Omega \mid z(L) \in T\}$. Each of these subgroups $L \in \Gamma'$ is uniquely determined by its intersection $T_L = T \cap L$, since L is the unique intrinsic SL_2-component in $C_G(z(T_L))$ containing T_L. Since $N_G(T)$ controls 2-fusion in T, $N_G(T)$ acts transitively on Γ'. Moreover, $T = T_L \times R_L$, where $R_L = C_T^\circ(L)$. Now set

$$\Gamma := \{L \leq G \mid L \cong (P)SL_2(\mathbb{F}), T \leq N_G(L) \text{ and } T \cap L \text{ is a torus in } L\}.$$

Note that $\Gamma' \subseteq \Gamma$.

LEMMA 9.3. *For any L, $M \in \Gamma$, L and M either commute or generate a simple algebraic group of type A_2, B_2, or G_2. Moreover, L and M are root $(P)SL_2$-subgroups of this algebraic group.*

Proof. If $p = \mathrm{pr}(T)$, then for any $L \in \Gamma$, the Prüfer 2-rank of $R_L = C_T^\circ(L)$ is $p - 1$. The standard "dimension" type of argument shows that for any $L, M \in \Gamma$,

$$\begin{aligned} p &= \mathrm{pr}(R_L R_M) \\ &= \mathrm{pr}(R_L) + \mathrm{pr}(R_M) - \mathrm{pr}(R_L \cap R_M) \\ &= (p-1) + (p-1) - \mathrm{pr}(R_L \cap R_M). \end{aligned}$$

Thus $\mathrm{pr}(R_L \cap R_M) = p - 2 \geq 1$. Let t be an involution in $R_L \cap R_M$. Then L, $M \leq E(C_G(t))$ by the B-conjecture. If they belong to different components, then they commute. Otherwise the component A that contains both L and M is a simple algebraic group, and moreover $T \cap A$ is a maximal torus (in the classical sense) in A. Hence, the lemma follows from Fact 3.1. ∎

Notation. For any $L \in \Gamma$, $N_L(T)T/T = N_L(T_L)/T_L$ is the Weyl group of SL_2 and has order 2; it contains a single involution, which will be denoted by r_L.

Note that the classical subgroup L is uniquely determined by r_L, since $T_L = [T, r_L]$ and $R_L = C_T^\circ(r_L)$.

LEMMA 9.4. *For any $L \in \Gamma'$ and $M \in \Gamma$, $[L, M] = 1$ if $[r_L, r_M] = 1$.*

Proof. It suffices to check this statement in the K-group $Y = \langle L, M \rangle$. Assume, without loss of generality, that Y is a simple algebraic group. Commuting reflections in the Weyl group of Y correspond to perpendicular roots. Note that the sum of two perpendicular roots is longer than both of them, hence if one of them is a long root, then the sum cannot be a root. ∎

9.3. *Weyl Group*

Denote $W := N_G(T)/C_G(T)$. The involutions r_L, for $L \in \Gamma$, belong to W and generate a subgroup W_0.

Now the aim is to construct a $\mathbb{Z}$-lattice on which W_0 acts as a crystallographic reflection group. First consider the elementary abelian p-subgroups E_p generated in T by all elements of the fixed prime order p. For the sake of complete reducibility of the action of W on E_p, one can consider only $p > |W|$.

Now note that $[E_p, r_L]$ is generated by a p-element in T_L and thus has order p. Hence E_p is a finite-dimensional vector space over $\mathbb{F}_p$ on which W_0 acts as a group generated by reflections (though it is not clear yet whether this action is faithful or not).

LEMMA 9.5. *The group W acts irreducibly on E_p.*

Proof. Note that $W_X = N_X(T)T/T$, being the Weyl group of a simple algebraic group, is generated by the involutions r_L which lie in it. Since T is a maximal torus in KX (or KK^*X, if $|V(K)| = 2$), $E'_p = E_p \cap T_X = [E_p, W_X]$ has codimension ≤ 2 in E_p, and W_X acts on E'_p irreducibly. Note that X has Lie rank at least 2, and hence the dimension of E'_p over $\mathbb{F}_p$ is at least 2. Finally, since T_K (and T_{K^*}, if it is in the picture) is conjugate by an element of W to a torus in T_X, the result follows. ∎

LEMMA 9.6. *The group W_0 acts irreducibly on E_p.*

Proof. If not, then, since $W_0 \lhd W$, E_p is the direct sum of W_0-invariant subspaces permuted by W, say $E_p = \oplus_{i=1}^m U_i$, where $m \ge 2$. Since W_X acts on E'_p irreducibly, E'_p projects isomorphically into one of U_i's, say U_1. Hence

$$\dim(U_1) = \cdots = \dim(U_m) \ge \dim(E'_p) \ge 2.$$

On the other hand, $\operatorname{codim}_{E_p}(E'_p) \le 2$ and hence $\sum_{i=2}^m \dim(U_i) \le 2$. Therefore $\dim E'_p = 2$, $m = 2$, $U_1 = E'_p$, and $E_p = E'_p \oplus E''_p$ after setting $U_2 = E''_p$.

It is easy to see that this means $W_0 = W_X \times W_Y$, where W_Y acts as a reflection group on E''_p and is conjugate to W_X in , $W_Y = W_X^w$. Taking some preimage n of w in $N_G(T)$, one can see that $Y = X^n$ contains K and commutes with X. This leaves us with the only possibility that $|V(K)| = 2$ and $C_Y^\circ(z) = KK^*$. But then for any classical subgroup $L < X * Y$ all classical subgroups commuting with L belong to $X * Y$, which contradicts the connectedness of the graph $\mathscr{D}$. ∎

9.4. *Root System*

The aim of this subsection is to construct a root system on which W_0 acts as a crystallographic reflection group. We need the following lemma for such a construction.

Throughout this subsection, let p be a prime number and E_{p^k} be the subgroup of T generated by elements of order p^k.

LEMMA 9.7. *Let k be a positive integer and $N = N_G(T)$. If $p^k \neq 2$, then $C_N(E_{p^k}) = C_G(T)$.*

Proof. It is clear that $C_G(T) \le C_N(E_{p^k})$. To see the converse, let $x \in C_N(E_{p^k})$. Since $x \in N$, it acts on the elements of Γ by conjugation.

First let us prove that x normalizes each subgroup in Γ. To get a contradiction, assume that there is some subgroup $L \in \Gamma$ such that $L^x \neq L$. But then by Lemma 9.3 L and L^x either commute or generate a semisimple group as root $(P)SL_2$-subgroups. Hence $|L \cap L^x| \le 2$. But this gives a contradiction since $L \cap E_{p^k} = L^x \cap E_{p^k} \le L \cap L^x$.

Hence for each $L \in \Gamma$, $L^x = L$ and x acts on $T \cap L$ as an element from $N_L(T \cap L)$, since $(P)SL_2$ does not have any outer automorphisms. Note that the Weyl group of $(P)SL_2$ is generated by an involution which inverts the torus $T \cap L$. Since x centralizes $E_{p^k} \cap T$, x centralizes $T \cap L$ for each $L \in \Gamma$, and hence x centralizes $T = \langle T \cap L \mid L \in \Gamma \rangle$ and $x \in C_G(T)$. This proves the equality. ∎

LEMMA 9.8. *There exists an irreducible root system on which W_0 acts as a crystallographic reflection group.*

Proof. First observe that, by Lemma 9.7, the kernel Z_0 of the representation of W_0 on E_p does not depend on the choice of prime $p > 2$ and that W_0 acts on E_{2^2} with the same kernel.

Let p be a prime number. Consider the sequence of subgroups

$$E_p \leftarrow E_{p^2} \leftarrow E_{p^3} \leftarrow \cdots$$

linked by the homomorphism $x \mapsto x^p$. The projective limit of this sequence is the free module V_p over the ring $\mathbb{Z}_p$ of p-adic integers. The action of W_0/Z_0 can be lifted to V_p, where it is still a faithful irreducible reflection group. By construction, V_p/pV_p is isomorphic to E_p as a W_0-module. Note also that W_0/Z_0 acts on the tensor product $V_p \otimes_{\mathbb{Z}_p} \mathbb{C}$ as a (complex) reflection group and that the dimension over $\mathbb{C}$ of $V_2 \otimes_{\mathbb{Z}_2} \mathbb{C}$ coincides with the Prüfer 2-rank of T, hence is at least 3.

Now by Fact 6.1 the quotient group W_0/Z_0 is one of the crystallographic reflection groups A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4 and acts on the corresponding root system. ∎

Let now $R = \{\bar{r}_i \mid i \in I\}$ be a simple system of reflections in $\overline{W}_0 = W_0/Z_0$. We shall identify I with the set of nodes of the Dynkin diagram for $\overline{W}_0$. It is well known that every reflection in an irreducible reflection group $\overline{W}_0$ is conjugate to a reflection in R.

LEMMA 9.9. *Every reflection $r \in W_0$ has the form r_M for some $(P)SL_2$-subgroup $M \in \Gamma$.*

Proof. Let $r \in W_0$ be a reflection. Working our way back through the construction of the module V_2, one can easily see that the Prüfer rank of $[T, r]$ is 1.

If now $r_L \in W_0$ is a reflection which corresponds to a classical subgroup $L \in \Gamma'$, then, comparing the Prüfer ranks of the groups $C_T(r_L)$ and $C_T(r)$, we see that $Z = (C_T(r_L) \cap C_T(r))^\circ$ has Prüfer rank at least 1. Hence the subgroup $\langle L, T, r\rangle$ contains a non-trivial central 2-torus; also note that $\langle L, T, r\rangle$ is a K-group. It is well known that a finite irreducible reflection group contains at most two conjugacy classes of reflections. Therefore, after replacing r and r_L with their appropriate conjugates, we can assume without loss of generality that the images of r_L and r in $\overline{W}_0$ correspond to adjacent nodes of the Dynkin diagram. Now we can easily see that $\langle L, T, r\rangle = Y * Z$ for some simple algebraic group Y of Lie rank 2 and that $r = r_M$ for some root $(P)SL_2$-subgroup of Y such that $M \in \Gamma$. ∎

9.5. *Final Step*

The next task is to prove that the conditions of the Lyons Theorem (Fact 5.2) are satisfied. For that purpose, the following lemma is helpful.

LEMMA 9.10. *Attach a $(P)SL_2$-subgroup $L_i \in \Gamma$ to each vertex of the Dynkin diagram I in such a way that the simple reflection $\bar{r}_i$ corresponding to the vertex is the image of r_{L_i} in $\overline{W}_0$. Then the following statements hold.*

(1) $[L_i, L_j] = 1$ *if and only if* $|\bar{r}_i\bar{r}_j| = 2$.

(2) $\langle L_i, L_j\rangle$ *is centrally isomorphic to* PSL_3 *if and only if* $|\bar{r}_i\bar{r}_j| = 3$.

(3) $\langle L_i, L_j\rangle$ *is centrally isomorphic to* PSp_4 *if and only if* $|\bar{r}_i\bar{r}_j| = 4$.

(4) L_i *and* L_j *are embedded in* $\langle L_i, L_j\rangle$ *as root* $(P)SL_2$*-subgroups.*

Proof. It is well known that for each $i, j \in I$ the order $|\bar{r}_i\bar{r}_j|$ takes the value 2, 3, or 4, depending on the number of edges in the Dynkin diagram connecting the vertices i and j. By Lemma 9.3, L_i and L_j either commute or generate $(P)SL_3$ or $(P)Sp_4$. The "only if" parts of (1) and (2) are easy to see.

In the case of Part (3), that is when L_i and L_j generate $(P)Sp_4$, we have to show that $|\bar{r}_i\bar{r}_j| \neq 2$. To get a contradiction, assume that L_i and L_j generate $(P)Sp_4$ and $|\bar{r}_i\bar{r}_j| = 2$. But then L_i and L_j are both short root SL_2-subgroups. Note that $\bar{r}_i$ and $\bar{r}_j$ are simple reflections, and it can be checked by inspection that one of them must be a long reflection. This proves the "only if" part of (3). Now Parts (1), (2), and (3) follow from Lemma 9.3 and the previous discussion. Part (4) is a direct consequence of Lemma 9.3. ∎

Finally we are in a position to apply the Lyons Theorem. Set G_0 to be the subgroup of G generated by the subgroups L_i for $i \in I$. (Note that $\langle K^*, X \rangle \leq G_0$.) By the Lyons Theorem, G_0 is a simple algebraic group over $\mathbb{F}$ with the Dynkin diagram I. Its Weyl group, with respect to the torus T, is W_0, hence it contains all subgroups from Γ, in particular $K \leq G_0$. By its construction, G_0 contains all subgroups in Ω commuting with K. Since all classical subgroups in $\Omega \cap G_0$ are conjugate in G_0, the same is true for any classical subgroup $L \in \Omega \cap G_0$. The connectedness of $\mathscr{D}$ implies that G_0 contains Ω, hence $G_0 = G$. This completes the proof of Theorem 1.3.

ACKNOWLEDGMENT

This paper would not have appeared without Alexandre Borovik, who introduced me to the problem and came up with lots of wonderful ideas for the proof. I also thank the referee for improving the text by making valuable comments.

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